Homoclinic Phenomena

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1 Introduction

Homoclinic orbits (or motions) were first defined by Poincaré in his treatise on the *restricted three body problem* [Poi87]. Further advances were made by Birkhoff [Bir60] in the 1930's, and, by Smale in the 1960's. Since that time they have been studied by many people and have been shown to be intimately related to our understanding of non-linear dynamical systems. There are many systems which possess homoclinic orbits. In one striking example (as discussed in the book of J. Moser [Mos73]), they can be used to account for the unbounded oscillatory motion discovered by Sitnikov in the three body problem. They also commonly occur in two dimensional mappings derived from periodically forced oscillations (e.g. see the book by Guckenheimer and Holmes [GH83]).

Roughly speaking a homoclinic orbit is an orbit of a mapping or differential equation which is both forward and backward asymptotic to a periodic orbit which satisfies a certain non-degeneracy condition called *hyperbolicity*. On its own such an orbit is only of mild interest. However, these orbits induce quite interesting structure among nearby orbits, and this latter fact is responsible for the main importance of homoclinic orbits. In addition, when homoclinic orbits are created in a parametrized system, many interesting and unexpected phenomena arise.

In this article we first describe the history and basic properties of homoclinic orbits. Next, we consider some simple polynomial diffeomorphisms of the plane (the so-called Hénon family) which exhibit homoclinic orbits. Subsequently, we discuss a general theorem due to Katok which gives sufficient conditions for the existence of such orbits. Finally, we briefly consider issues related to homoclinic bifurcations and some of their consequences.

2 Homoclinic Orbits in Diffeomorphisms

Consider a discrete dynamical system given by a C^r diffeomorphism $f : M \to M$ where M is a C^∞ manifold and r is a positive integer. That is, f is bijective and both f and f^{-1} are $r-$times continuously differentiable. Given a point $x \in M$, set $x_0 = x$. For non-negative integers n we inductively define $x_{n+1} = f(x_n)$ and $x_{-n-1} = f^{-1}(x_{-n})$. We also write $f^n(x) = x_n$ for n in the set $\mathbf{Z}$ of all integers. The *orbit* of x is the set $O(x) = \{f^n(x) : n \in Z\}$.

A *periodic point* p of f is a point such that there is a positive integer $N > 0$ such that $f^N(p) = p$. The least such number $\tau(p)$ is called the *period* of p. If $\tau(p) = 1$, we call p a *fixed point*. The periodic point p with period τ is called called *hyperbolic* if all eigenvalues of the derivative $Df^\tau(p)$ at p have absolute value different from 1. For convenience, we refer to the eigenvalues of $Df^\tau(p)$ as *eigenvalues associated to* p. If p is a hyperbolic periodic point all of whose associated eigenvalues have norm less than one, we call p a *sink* or *attracting periodic point*. The opposite case in which all associated eigenvalues have norm larger than one is called a *source*. A hyperbolic periodic point p which is neither a source nor sink is called a *saddle* or *hyperbolic saddle*.

Given a saddle p of period τ, we consider the set $W^s(p) = W^s(p, f)$ of points $y \in M$ which are forward asymptotic to p under the iterates $f^{n\tau}$. That is, the points $y \in M$ such that $f^{n\tau}(y) \to p$ as $n \to \infty$. This is called the *stable set* of p. Similarly, we consider the *unstable set* of p which we may define as $W^u(p) = W^u(p, f) = W^s(p, f^{-1})$. The Stable Manifold Theorem guarantees that $W^s(p)$ and $W^u(p)$ are injectively immersed submanifolds of M whose dimensions add up to *dim* M. In these cases, they are called the *stable* and *unstable* manifolds of p, respectively. A point $q \in W^s(p) \bigcap W^u(p) \setminus \{p\}$ is called a *homoclinic point* of p (or of the pair (f, p)). If the submanifolds $W^s(p)$ and $W^u(p)$ meet transversely at q, then q is called a *transverse homoclinic point*. Otherwise, q is called a *homoclinic tangency*.

In the special case when M is a two-dimensional manifold, the stable and unstable manifolds of a saddle periodic point p are injectively immersed curves in M. A transverse homoclinic point q of p is a point of intersection off p where the curves are not tangent to each other. This is depicted in Figure 1 for the case of a saddle fixed point for the map $H(x, y) = (7 - x^2 - y, x)$

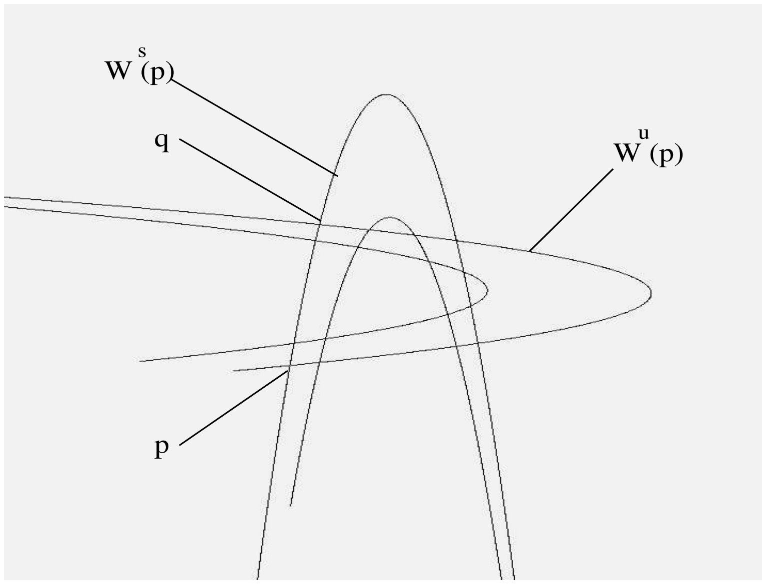

Figure 1: Stable and unstable manifolds in the map $H(x, y) = (7 - x^2 - y, x)$ for the fixed point $p \approx (-3.83, -3, 83)$.

which is a member of the so-called Hénon family which we will discuss later. The figure was made using the numerical package *Dynamics* which comes with the book [NY98] of H. E. Nusse and J. A. Yorke.

One easily sees that every point in the orbit of a transverse homoclinic point q of a hyperbolic saddle fixed point p is again a transverse homoclinic point of p. Also, the curves $W^u(p)$ and $W^s(p)$ are *invariant*; i.e., $f(W^u(p)) = W^u(p)$ and $f(W^s(p)) = W^s(p)$. This implies that the curves $W^u(p)$ and $W^s(p)$ extend, wind around, and accumulate on each other forming a complicated web.

Upon seeing this complicated structure in the restricted three body problem, Poincaré very poetically wrote (p. 389, [Poi87])

> Que l'on cherche à se représenter la figure formée par ces deux courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées; chacune des deux courbes ne doit jamais se recouper elle-même, mais elle doit se replier sur elle-même d'une manière

> trés complexe pour venir recouper une infinité de fois toutes les mailles du réseau.
>
> On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous les problèmes de Dynamique où il n'y a pas d'intégrale uniforme . . .

The next major advance concerning homoclinic orbits was made by Birkhoff [Bir60] who proved that in every neighborhood of a transverse homoclinic point of a surface diffeomorphism one can find infinitely many distinct periodic points. Birkhoff also presented a symbolic description of the nearby orbits and noticed the analogy with Hadamard's description of geodesics on a surface. Birkhoff's analysis was generalized by Smale to arbitrary dimension, and, in addition, Smale gave a simpler analysis of the associated nearby orbits in terms of compact zero-dimensional symbolic spaces which we now call *shift spaces* or *topological Markov chains*.

Once one knows that a diffeomorphism f has a transverse homoclinic point for a saddle periodic point p, it is interesting to consider the closure of the orbits of all such homoclinic points. This turns out to be a closed invariant set containing a dense orbit and a countable dense set of periodic saddle points [New80]. It is usually called a *homoclinic closure* or $h-$closure. These sets form the basis of chaotic or irregular motions in non-linear systems.

3 The Smale Horse-shoe map and Associated Symbolic System

To understand the geometric picture discovered by Smale, it is best to start with a concrete example of a diffeomorphism of the plane known as the *Smale horse-shoe diffeomorphism.*

Given any homeomorphism $f : X \to X$ on a space X and a subset $U \subset X$, let us define $I(f, U)$ to be the set of points $x \in X$ such that $f^n(x) \in U$ for every integer n. Thus, we have

$$I(f, U) = \bigcap_{n \in \mathbf{Z}} f^n(U).$$

We call $I(f, U)$ the *invariant set* of f in U, or, alternatively, the invariant set of the pair (f, U).

We now construct a special diffeomorphism f of the Euclidean plane to itself in which $U = Q$ is the unit square and for which $I(f, U)$ has a very interesting structure. It is this map which is usually known as the Smale horse-shoe map.

Let $Q = [0, 1] \times [0, 1]$ be the unit square in the plane $\mathbf{R}^2$. Let $0 < \alpha < \frac{1}{2}$, and consider a diffeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ which is a composition of two diffeomorphisms $f = T_2 \circ T_1$ as follows. The map $T_1(x, y) = (\alpha^{-1}x, \alpha y)$ contracts vertically, expands horizontally, and maps Q to the thin rectangle $Q_1 = \{(x, y) : 0 \le x \le \alpha^{-1},\ 0 \le y \le \alpha\}$ which is short and wide. The map T_2 bends the right side of Q_1 up and around so that $T_2(Q_1) = f(Q)$ has the shape of a "horse-shoe" or "rotated arch." We arrange for T_2 to take the lower right corner of Q_1 up to the upper left corner of Q in such a way that $f(Q)$ meets Q in two full width subrectangles which we call R_1 and R_2. This can be done in such a way that the pre-images $R_1^{-1} = T_1^{-1}(R_1)$ and $R_2 = T_1^{-1}(T_2^{-1}(R_2))$ are both full-height subrectangles of Q, and the restricted maps $f_1 \stackrel{\text{def}}{=} f \mid R_1^{-1}$ and $f_2 \stackrel{\text{def}}{=} f \mid R_2^{-1}$ are both affine. Thus, we arrange that f_1 is simply the restriction of T_1 to R_1^{-1}, and the map f_2 can be expressed in formulas as $f_2(x, y) = (-\alpha^{-1}x + \alpha^{-1}, -\alpha\ y + 1)$. This construction implies that f will have the origin $p = (0, 0)$ as a hyperbolic fixed point. We label the upper left corner $(0, 1)$ of Q with the letter q. It follows that the bottom and left edges of Q will be in the unstable and stable manifolds of p, respectively, and we have indicated this in Figure 2 with small arrows.

The above construction gives us a diffeomorphism f of the plane $\mathbf{R}^2$ such that $Q_1^+ \stackrel{\text{def}}{=} f(Q) \bigcap Q = R_1 \bigcup R_2$ is the union of two full-width subrectangles of Q. We wish to describe $I(f, Q)$. We begin with the sets $Q^+ = \bigcap_{n \ge 0} f^n(Q)$ and $Q^- = \bigcap_{n \ge 0} f^{-n}(Q)$. Thus, Q^+ is simply the set of points in Q whose backward orbits stay in Q, and Q^- is the set of points whose forward orbits stay in Q. For $i = 1, 2$, each rectangle R_i is mapped to a thin horse-shoe in $f(Q)$ which meets Q in two full-width subrectangles. Combining these for $i = 1, 2$ gives four full-width rectangles as shaded in Figure 3. Thus, $Q \bigcap f(Q) \bigcap f^2(Q)$ consists of these four subrectangles. Figure 3 shows the sets $f^2(Q), f^{-2}(Q)$ as well as the shaded rectangles we just mentioned.

Continuing in this way, one sees that, for each $n > 0$, the set $Q_n^+ = Q \bigcap f(Q) \bigcap \ldots \bigcap f^n(Q)$ consists of 2^n full-width subrectangles of Q, each with

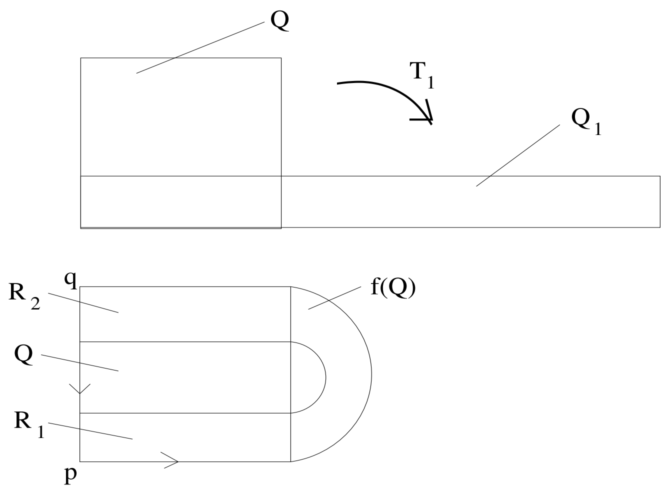

Figure 2: The horse-shoe map

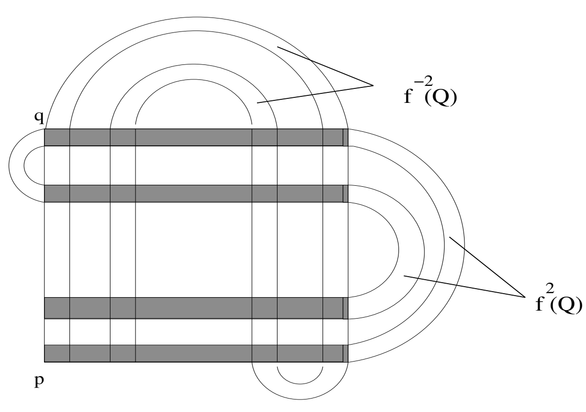

Figure 3: The sets $f^2(Q)$ and $f^{-2}(Q)$ for the horse-shoe map f.

height α^n. It follows that $Q^+ = \bigcap_n f^n(Q)$ is an interval cross a Cantor set. Analogously, Q^- is a Cantor set cross an interval, and the set $I(f, Q)$ is a Cantor set in the plane. Let us recall the definition of a Cantor set C in a metric space X. We first define a Cantor space C to be a compact, perfect, totally disconnected metric space. That is, C is a compact metric space, whose connected components are points such that every point x in C is a limit point of $C \setminus \{x\}$. A Cantor set C in a metric space X is a subset which is a Cantor space in the induced subspace (relative) topology.

The dynamics of f on the invariant set $I(f, Q)$ can be conveniently described as follows.

Let $\Sigma_2 = \{1, 2\}^{\mathbf{Z}}$ be the set of doubly infinite sequences of one's and two's. Writing elements $\mathbf{a} \in \Sigma_2$ as $\mathbf{a} = (a_i) = (a_i)_{i \in Z}$, we define a metric ρ on Σ_2 by

$$\rho(\mathbf{a}, \mathbf{b}) = \sum_{n \in \mathbf{Z}} \frac{1}{2^{| n |}} | a_i - b_i |.$$

The pair (Σ_2, ρ), then, is a Cantor space.

The *left shift automorphism* on Σ_2 is the map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(\mathbf{a})_i = a_{i+1}$ for each $i \in \mathbf{Z}$. This is a homeomorphism from Σ_2 to itself. It has a dense orbit and a dense set of periodic points.

For a point $x \in I(f, Q)$, define an element $\phi(x) = \mathbf{a} = (a_i) \in \Sigma_2$ by $a_i = j$ if and only if $f^i(x) \in R_j$. It turns out that the map $\phi : I(f, Q) \to \Sigma_2$ is a homeomorphism such that $\sigma\phi = \phi f$.

In general, given two discrete dynamical systems $f : X \to X$, and $g : Y \to Y$, a homeomorphism $h : X \to Y$ such that $gh = hf$ is called a topological conjugacy from the pair (f, X) to the pair (g, Y). When such a conjugacy exists, the two systems have virtually the same dynamical properties.

In the present case, one sees that the dynamics of f on $I(f, Q)$ is completely described by that of σ on Σ_2.

It turns out the the Smale horse-shoe map contains essentially all of the geometry necessary to describe the orbit structures near homoclinic orbits. To begin to see this, recall that the left and bottom boundaries of Q were in the stable and unstable manifolds of p. Extending these curves as in Figure 4, one sees that the three corners of Q different from p are, in fact, all transverse homoclinic points of p.

It was a great discovery of Smale that, in the case of a general transverse homoclinic point, one sees the above geometric structure after taking some power f^N of the diffeomorphism f. Thus, we have

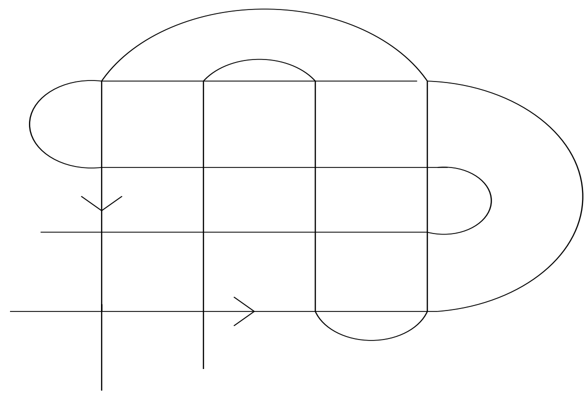

Figure 4: Stable and Unstable manifolds in the horse-shoe map

Theorem 3.1 (Smale). *Let $f : M \to M$ be a C^1 diffeomorphism of a manifold M with a hyperbolic periodic point p and a transverse homoclinic point q of the pair (f, p). Then, one can find a positive integer N and a compact neighborhood U of the points p and q such that the pair $(f^N, I(f^N, U))$ is topologically conjugate to the full 2-shift (σ, Σ_2).*

In modern language, we can assert that more is true. Let $\Lambda(f) = \bigcup_{0 \le j < N} f^j(I(f^N, U))$ be the $f-$orbit of the set $I(f^N, U)$. Then, $\Lambda(f)$ is a compact zero-dimensional hyperbolic basic set for f with $V \stackrel{\text{def}}{=} \bigcup_{0 \le j < N} f^j(U)$ as an *adapted* or *isolating* neighborhood. This means that $\Lambda(f) = \bigcap_{n \in \mathbf{Z}} f^n(V)$ is a compact, zero-dimensional hyperbolic set (see [Rob99] for definitions and related references) contained in the interior of V and $f \mid \Lambda(f)$ has a dense orbit. If g is C^1 near f then $\Lambda(g) \stackrel{\text{def}}{=} \bigcap_{n \in \mathbf{Z}} g^n(V)$ is a hyperbolic basic set for g and the pairs $(f, \Lambda(f))$ and $(g, \Lambda(g))$ are topologically conjugate.

To get some appreciation for the magnitude of the contribution here, one might note the complicated arguments employed by Poincaré at the end of [Poi87] to show that so-called heteroclinic points (intersections between stable and unstable manifolds of saddles with different orbits) existed. Birkhoff found a symbolic description (using a infinitely many symbols) of the orbits near a transverse homoclinic orbit from which the existence of both infinitely many periodic and heteroclinic points is obvious. Smale extended the treatment of transverse homoclinic points to all dimensions, and found the symbolic description (using two symbols for some iterate of the map) given above. Moreover, Smale proved the *robustness* of these structures:

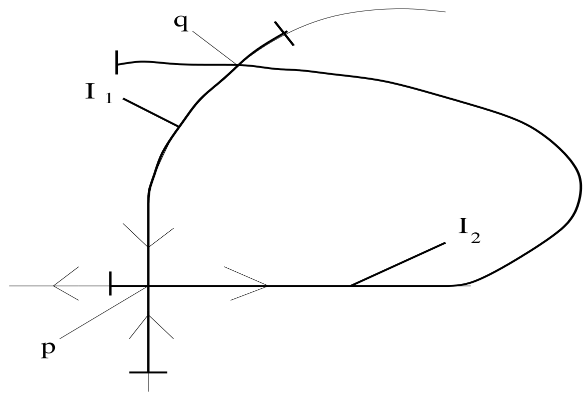

Figure 5: The curves $I_1 \subset W^s(p)$ and $I_2 \subset W^u(p)$

they persist under small C^1 perturbations. Note that Poincaré's discovery of homoclinic points was in 1899, Birkhoff's results came in 1935, and Smale's results came in 1965. Thus, the above advances took over 65 years!

One can understand the geometry of Smale's construction fairly easily in the two dimensional case. Let q be the transverse homoclinic point of the saddle fixed point p of the C^r diffeomorphism f on the plane $\mathbf{R}^2$. Given a small neighborhood $\tilde{U}$ of p, let $W^s(p, \tilde{U})$ denote connected component of $W^s(p) \bigcap \tilde{U}$ containing p, and define $W^u(p, \tilde{U})$ similarly. We may choose C^r coordinates (x, y) so that in some small neighborhood $\tilde{U}$ of p, the point p corresponds to $(0, 0)$, the set $W^u(p, \tilde{U})$ corresponds to $(y = 0)$, and the set $W^s(p, \tilde{U})$ corresponds to $(x = 0)$. We assume that $\tilde{U}$ is small enough that f in $\tilde{U}$ is closely approximated by its derivative $Df_{(0,0)}$. Hence, f nearly contracts vertical directions and expands horizontal directions in $\tilde{U}$.

Take compact arcs $I_1 \subset W^s(p)$ and $I_2 \subset W^u(p)$ both containing the points p and q as in Figure 5.

Let D be a curvilinear rectangle which is a slight thickening of I_1. The forward iterates $f^i(D)$ will stay near I_1 for awhile and then start to approach I_2. If we choose D appropriately, we can arrange for some high iterate $f^N(D)$ to be a slight thickening of I_2 as in Figure 6. This looks geometrically like the horse-shoe map. Let A_1 be the connected component of the intersection $D \bigcap f^N(D)$ containing p, and let A_2 be the connected component of the intersection $D \bigcap f^N(D)$ containing q. These sets (which are shaded in Figure 6) play the role of the rectangles R_1 and R_2, respectively, in the horse-shoe construction. We use the set $A_1 \bigcup A_2$ for U in Theorem 3.1.

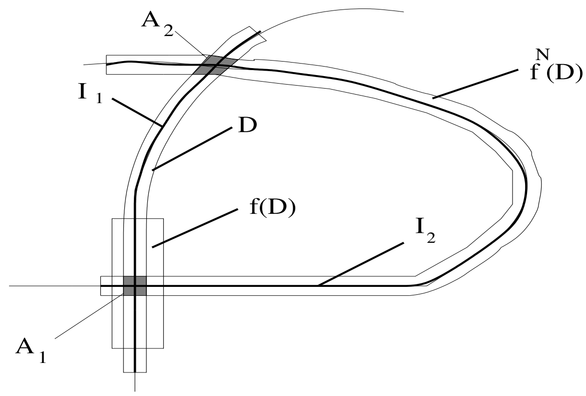

Figure 6: The curvilinear rectangle D and its N-th iterate $f^N(D)$ are geometrically like the horse-shoe map.

4 The Hénon family

To give explicit formulas for the horse-shoe map above is somewhat tedious, and it is of interest to note that similar properties occur in maps with simple formulas. Indeed, such properties occur quite often in a well-known family of maps known as the *Hénon family.* As we have mentioned, the map in Figure 1 provides an example.

One may simply define a Henon map as a diffeomorphism $H = (H_1(x, y), H_2(x, y))$ with inverse $G(x, y) = (G_1(x, y), G_2(x, y))$ such that all the maps $F_i(x, y), G_i(x, y)$ are polynomials of degree at most two. It is known (see e.g. [FM89]) that such maps H have constant Jacobian determinant, and, up to affine conjugacy, may be represented in the form $H = H_{a,b}(x, y) = (a - x^2 - by, x)$ with a, b constants and $b \neq 0$. This makes sense when all the terms are real or complex. In the real case, we speak of the *real Henon family* and, in the complex case, we speak of the *complex Henon family.*

The real Henon family was first presented by the physicist M. Hénon in 1976 as perhaps the simplest non-linear diffeomorphism of the plane exhibiting a so-called *strange attractor.* These mappings in the real and complex cases have been the focus of much attention. Our interest here is that, at least for certain parameters a, b, they provide concrete globally defined maps whose dynamics are analogous to that of the horse-shoe diffeomorphism. In fact, Devaney and Nitecki [DN79] proved (in the real case) that for fixed $b \neq 0$, there is a constant $a_0 > 0$ such that if $a > a_0$, then the set $B_{a,b}$

of bounded orbits of $H_{a,b}$ is a compact zero-dimensional set and the pair $(H_{a,b}, B_{a,b})$ is topologically conjugate to (σ, Σ_2). In addition, it can be shown that the invariant set $B_{a,b}$ is a single hyperbolic $h-$closure. Analogous results are true for the complex Henon family and proofs were originally given in the thesis of Ralph Oberste-Vorth (unpublished) under the supervision of John Hubbard at Cornell University. More recent proofs are in [New04] and [Hru04]. Many interesting results have been obtained for the complex Hénon map by Bedford and Smillie and Sibony and Fornaess (see the references in [Hru04]).

5 Homoclinic points in systems with positive topological entropy

There is an invariant of topological conjugacy which is known as the topological entropy. In a certain sense, this gives a quantitative measurement of the *amount* of complicated or chaotic motion in the system.

Let $f : X \to X$ be a continuous self-map of the compact metric space (X, d). For a positive integer $n > 0$, we define an $n-$orbit to be a finite sequence $O(x, n) = \{x, f(x), \dots, f^{n-1}(x)\}$. Given a positive real number $\epsilon > 0$, we say that two $n-$orbits $O(x, n)$ and $O(y, n)$ are *ϵ-distinguishable* if there is a $0 \le j < n$ such that $d(f^j x, f^j y) > \epsilon$. Another way to look at this is the following. Define the so-called d_n-metric on X by setting $d_n(x, y) = max_{0\le j<n} d(f^j x, f^j y)$. Then, the two $n-$orbits $O(x, n), O(y, n)$ are $\epsilon-$distinguishable if and only if $d_n(x, y) > \epsilon$. It follows from compactness of X and the uniform continuity of each of the maps f^j, $0 \le j < n$, that the number $r(n, \epsilon, f)$ of $\epsilon-$distinguishable $n-$orbits is finite for each given $\epsilon > 0$ and each positive integer n. We define the number

$$h(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(n, \epsilon, f).$$

This means that, for some sequence of integers $n_1 < n_2 < \dots$, the map f has roughly $e^{n_i h(f)}$ $\epsilon-$distinguishable n_i-orbits for i large and ϵ small.

The number $h(f)$ is called the *topological entropy* of the map f. It may be infinite for homeomorphisms, but it is always finite for smooth maps on finite dimensional manifolds. The number $h(f)$ has many nice properties. For instance, $h(f^N) = Nh(f)$ for every positive integer N, and, if f is a homeomorphism, then $h(f^{-1}) = h(f)$. Further, if f and g are topologically conjugate, then $h(f) = h(g)$. The so-called *Variational Principle for Topological Entropy* asserts that $h(f)$ is the supremum of the measure theoretic entropies of the invariant probability measures for f. Our interest in this invariant here is the following theorem of Katok.

Theorem 5.1 (Katok). *Let f be a C^2 diffeomorphism of a compact two dimensional manifold M to itself with positive topological entropy. Then, f has transverse homoclinic points.*

If fact, Katok extended this theorem (see the supplement in [HK95]) to show that, if $h(f) > 0$ and $\epsilon > 0$, then there is a compact zero-dimensional hyperbolic basic set Λ for h such that $h(f, \Lambda) > h(f) - \epsilon$. Thus, one can find nice invariant topologically transitive sets for f (i.e., sets with dense orbits) on which the topological entropies of restriction of f are arbitrarily close to that of f.

This theorem has the interesting consequence that the map $f \to h(f)$ is lower-semi-continuous on the space of C^2 diffeomorphisms of a surface. It was proved in [New89] (and, independently by Yomdin in [Yom87]) that the map $f \to h(f)$ is upper-semi-continuous on the space of C^∞ diffeomorphisms of any compact manifold. Combining these results gives the theorem that the map $f \to h(f)$ is continuous on the space of C^∞ diffeomorphisms on a compact surface, and that positivity of $h(f)$ implies the existence of transverse homoclinic points.

It is also worth noting that, for any continuous self-map $f : M \to M$ on a compact manifold M, one has the inequality $h(f) \geq \log \mid \mu \mid$ where μ is the eigenvalue of largest norm of the induced map $f_\star$ on the first real homology group [Man75]. Putting this together with Theorem 5.1 gives the fact that *there are whole homotopy classes of diffeomorphisms on surfaces all of whose elements have transverse homoclinic points*. For instance, consider a 2×2 matrix $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries, determinant one, and eigenvalues λ_1, λ_2 with $0 < \mid \lambda_1 \mid < 1 < \mid \lambda_2 \mid$. Let $\tilde{L} : \mathbf{T}^2 \to \mathbf{T}^2$ be the induced diffeomorphism on the two-dimensional torus $\mathbf{T}^2$. This is an example of what is called an *Anosov* diffeomorphism. In this case the number μ above is simply λ_2, and this holds for any diffeomorphism f of $\mathbf{T}^2$ which can be continuously deformed into $\tilde{L}$. Hence, any such f must have transverse homoclinic points.

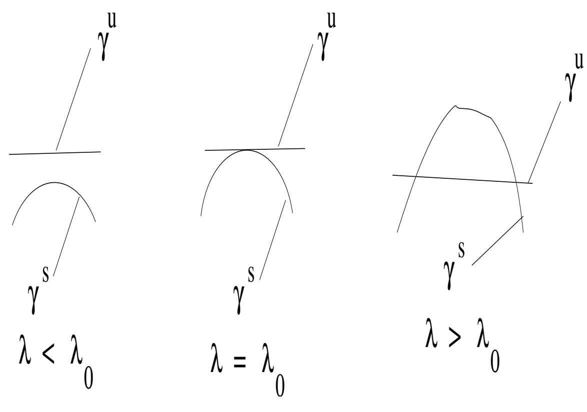

Figure 7: Creation of a homoclinic tangency

6 Homoclinic tangencies

Let $\{f_\lambda, \lambda \in [0,1]\}$ be a parametrized family of C^r diffeomorphisms of the plane with λ an external parameter. It frequently occurs that there is a hyperbolic saddle fixed point p_λ for each parameter λ moving continuously with λ such that a some value λ_0, a homoclinic tangency is created at a point q_0. This means that there are an $\epsilon > 0$, a small neighborhood U of q_0, and curves $\gamma^u_\lambda \subset W^u(p_\lambda)$, $\gamma^s_\lambda \subset W^s(p_\lambda)$ such that $\gamma^s_\lambda \bigcap \gamma^u_\lambda = \emptyset$ for $\lambda_0 - \epsilon < \lambda < \lambda_0$, $\gamma^s_{\lambda_0} \bigcap \gamma^u_{\lambda_0} = \{q_0\}$, and $\gamma^s_\lambda \bigcap \gamma^u_\lambda$ consists of two distinct points for $\lambda_0 < \lambda < \lambda + \epsilon$. In most cases, the tangency of $\gamma^u_{\lambda_0}$ and $\gamma^s_{\lambda_0}$ at q_0 will be of the second order, and we will assume that occurs here. The geometry is as in Figure 7.

The creation of homoclinic tangencies is part of the general subject of *homoclinic bifurcations*. A recent survey of this subject is in the book of Bonatti, Diaz, and Viana [BDV05]. Typical results are the following. If $p = p_{\lambda_0}$ is a saddle fixed point whose derivative is area-decreasing (i.e., $| \, Det(Df(p)) \, | < 1$), then there are infinitely many parameters λ near λ_0 for which *each transverse homoclinic point* of p_λ is a limit of periodic sinks (asymptotically stable periodic orbits) [New79], [Rob83]. In addition, so-called strange attractors and SRB-measures appear [MV93].

Finally, we mention that recently it has been shown that, generically in the C^r topology for $r \geq 2$, homoclinic closures associated to a homoclinic tangency (in dimension two) have maximal Hausdorff dimension (Theorem 1.6 in [DN05]).

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